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The spectrum of a turbulent passive scalar in the viscous–convective range

By J. QIAN

Department of Physics, Graduate School of Academia Sinica, P.O. Box 3908, Beijing 100039, China

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The closed equations of isotropic turbulence, obtained by the method of non-equilibrium statistical mechanics and a perturbation-variation approach (Qian 1983, 1985, 1988), are applied to the study of the spectrum dynamics of a turbulent passive scalar in the viscous–convective range. Batchelor's k^{-1} spectrum is further confirmed. Moreover the effective average value of the least principal rate of strain γ in Batchelor's spectrum function is theoretically evaluated and it is found that $\gamma^{-1} = C(\nu/\epsilon)^{\frac{1}{2}}$ with $C = 2\sqrt{5}$. Here ν is the kinematic viscosity, and ϵ is the energy dissipation rate. This prediction is in agreement with experimental data reported by Grant *et al.* (1968) and Williams & Paulson (1977).

1. Introduction

We study a passive scalar field convected by an isotropic turbulence with very high Prandtl number, so the Corrsin wavenumber $k_c = (\epsilon/\mu^3)^{\frac{1}{4}}$ is much greater than the Kolmogorov wavenumber $k_d = (\epsilon/\nu^3)^{\frac{1}{4}}$; here ϵ is the energy dissipation rate, μ is the diffusion coefficient and ν is the kinematic viscosity. In this case there exists a wavenumber range: $k_d \ll k \ll k_c$, which is called the viscous–convective range. In the viscous–convective range the viscosity plays a dominant role but the diffusion can be neglected. Batchelor (1959) has predicted that the scalar-variance spectrum in the viscous–convective range is

$$F(k) = \chi(\gamma k)^{-1}. \tag{1}$$

Here

$$\chi = 2\mu \int_0^\infty k^2 F(k)\,\mathrm{d}k \tag{2}$$

is the dissipation rate of scalar variance, γ is the effective average least principal rate of strain and

$$\gamma^{-1} = C(\nu/\epsilon)^{\frac{1}{2}}, \tag{3}$$

where C is an universal constant.

Much interest and effort have been directed to the determination of the universal constant C in (3). According to experimental data of the velocity derivative, Batchelor (1959) concluded that $C = 2$. Reid (1955) has suggested that $C = 2.5$. Gibson (1968) noted that $\sqrt{3} < C < 2\sqrt{3}$. Kraichnan (1968) pointed out that Batchelor's theory ignored the statistical character of the least principal rate of strain and the actual value of C could be much greater than the values suggested by Batchelor, Reid, and Gibson. Grant *et al.* (1968) have conducted experiments in the ocean to measure temperature and velocity fluctuation and found that $C = 3.9 \pm 1.5$. Kraichnan's abridged Lagrangian-history direct interaction (ALHDI) approximation has been applied to calculate C and gives $C < 0.9$, which is a poor prediction (see Leslie 1973). Williams & Paulson (1977) have made measurements of the temperature and velocity spectrum in the atmospheric boundary layer, and their data are in good agreement with the experimental data reported by Grant *et al.* (1968). By the test field model (TFM) Newman & Herring (1979) obtained $C = 1.68$ and 0.68, corresponding to two different choices of the adjustable empirical constants of the TFM; they think that $C = 1.68$ is near to experimental data and $C = 0.68$ close to Kraichnan's ALHDI estimate. Hence it is desirable and attractive to have a theoretical prediction of the universal constant C which agrees with experimental data reported by Grant *et al.* (1968) and is free of adjustable empirical constants.

In this paper the non-equilibrium-statistical-mechanics theory of isotropic turbulence (Qian 1983, 1985, 1988) is applied to derive Batchelor's k^{-1} spectrum, (1), (3), and in particular to evaluate the universal constant C in Batchelor's spectrum function. The prediction $C = 2\sqrt{5}$ obtained in this paper is in agreement with experimental data. Then some relevant problems are discussed.

2. Non-equilibrium statistical mechanics theory of turbulence

The method of non-equilibrium statistical mechanics combined with a perturbation-variation approach has been applied to solve the closure problem of turbulence theory and to calculate the velocity spectrum (Qian 1983, 1984). A brief description of this closure method is as follows. A complete set of independent real modal parameters, which are linear combinations of Fourier components of the velocity field, and its dynamic equations are worked out to describe the modal dynamics of a turbulent velocity field. According to non-equilibrium statistical mechanics, the probability density function of the turbulent velocity field satisfies the Liouville equation corresponding to the dynamic equation. A perturbation solution of the Liouville equation is obtained by using the Fokker–Planck (FP) operator to approximate the Liouville operator in the Liouville equation; the FP operator contains a dynamic damping coefficient. By using the perturbation solution and the dynamic equation, the energy equation is derived, which is similar to that of Kraichnan's direct-interaction approximation (DIA). The dynamic damping coefficient in the FP operator is treated as an optimum control parameter to minimize the error of the perturbation solution of the Liouville equation. By a variational approach, a convergent integral equation for the dynamic damping coefficient is obtained to replace the divergent response equation of Kraichnan's DIA, thereby solving the closure problem without appealing to a Lagrangian formulation. The resulting closed set of integral equations has been solved by the equation-error method to obtain the following velocity spectrum in the universal equilibrium range (inertial range and viscous range) (Qian 1984, 1987):

$$E(k) = \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} f(k/k_{\mathrm{d}}), \tag{4a}$$

$$f(x) = 1.19(1+5.3x^{\frac{2}{3}})\exp(-5.4x^{\frac{4}{3}}). \tag{4b}$$

Generally speaking turbulent scalar and velocity fields are interrelated and interact with each other. If the scalar is passive, i.e. its amplitude is small enough not to effect the velocity field, the velocity field can be considered to be independent of the passive scalar. Of course the action of the turbulent velocity field on the scalar field still plays a dominant role, in particular the equations of the scalar-variance spectrum will contain the velocity spectrum $E(k)$. The non-equilibrium-statistical-mechanics closure method described above has been extended to the study of a passive scalar field convected by turbulence (Qian 1985). First, a complete set of independent real modal parameters and their dynamic equations are formulated to describe the turbulent passive scalar field, and the corresponding Liouville equation is derived for the conditional probability density function of the turbulent scalar field, the velocity field being given. Then a perturbation solution of the Liouville equation is obtained by using the FP operator to approximate the Liouville operator; the FP operator contains a effective damping coefficient $\Omega(k)$. By using this perturbation solution the higher-order correlations of the turbulent scalar–velocity field are expressed in terms of their lower-order correlations; then, from the dynamic equation we obtain the following variance equation (Qian 1985):

$$(d_t + 2\mu k^2)\,F(k) = S(k), \tag{5a}$$

where

$$S(k) = 16\pi^2 k^4 \int_0^\infty \mathrm{d}p\, C(k,p)\, p^4 \frac{g(p)-g(k)}{\Omega(k)+\Omega(p)} \tag{5b}$$

is the scalar-variance transfer spectrum function, $g(k) = F(k)/(4\pi k^2)$ and

$$C(k,p) = (4\pi)^{-1} \int_0^\pi \mathrm{d}\phi \sin^3\phi E(r)\, r^{-4}, \quad r = |\boldsymbol{k}-\boldsymbol{p}|. \tag{6}$$

Here ϕ is the angle made by the wavevectors $\boldsymbol{k}$ and $\boldsymbol{p}$. The variance equation (5) implies the conservation of scalar variance. The effective damping coefficient $\Omega(k)$ is treated as an optimum control parameter to minimize the error of the perturbation solution of the Liouville equation. Finally, by a variation procedure we obtain the following integral equation for the effective damping coefficient $\Omega(k)$ (Qian 1985);

$$g(k)\,\Omega(k) = 4\pi k^2 \int_0^\infty \mathrm{d}p\, C(k,p)\, p^4 \Omega(p) \frac{g(k)-g(p)}{[\Omega(k)+\Omega(p)]^2}. \tag{7}$$

The variance equation (5), (6) and the Ω equation (7) constitute the closed set of equations for the spectrum dynamics of a turbulent passive scalar field.

Various closure methods yield nearly the same energy equation (or variance equation in the case of turbulent passive scalar), hence it has long been understood (Leslie 1973) that the crux of the matter is how to determine the damping coefficient or the relaxation time. The eddy-damped quasi-normal Markovian (EDQNM) approximation is a modification of the defective quasi-normal theory, but the modification is made *a posteriori* and lacks fundamental justification (Orszag 1974). The EDQNM approximation employs heuristic reasoning to determine the damping coefficient to within several adjustable empirical constants, and then find these constants through a comparison of results to experimental data or a comparison of the phenomenological EDQNM approximation to a self-consistent analytical method free of arbitrariness. On the other hand the approach proposed by this author (Qian 1983, 1985, 1988) is a self-consistent analytical method free of arbitrariness, which treats the damping coefficient as an optimum control parameter to minimize the error of the perturbation solution of the Liouville equation, leading to an integral equation for the damping coefficient which contains no empirical constants. Actually, according to modern control theory, the problem of determining the damping coefficient is just a optimum-parameter-estimation problem. Empirical constants of the EDQNM approximation could be determined by a comparison of the phenomenological EDQNM approximation to the self-consistent analytical method adopted in this paper.

3. The integral $C(k,p)$

Both the variance equation (5) and the Ω-equation (7) contain the integral $C(k,p)$, which is a functional of the velocity spectrum $E(k)$ and represents the action of the turbulent velocity field upon the passive scalar field. Letting $Y=\cos\phi$ and substituting (4a) into (6), we have

$$C(k,p) = (4\pi)^{-1}\epsilon^{\frac{2}{3}}k_{\mathrm{d}}^{-\frac{17}{3}}\tilde{C}, \tag{8a}$$

$$\tilde{C} = \int_{-1}^{1}\mathrm{d}Y(1-Y^2)\,f(z)\,z^{-\frac{17}{3}}, \quad z = r/k_{\mathrm{d}}, \tag{8b}$$

where $r=(k^2+p^2-2Ykp)^{\frac{1}{2}}$. Since $-1 \leqslant Y \leqslant 1$, we have

$$A \leqslant z \leqslant B, \quad A = |k-p|/k_{\mathrm{d}}, \quad B = (k+p)/k_{\mathrm{d}}. \tag{8c}$$

By using z as the dummy variable of integration, after some manipulation (8b) becomes

$$\tilde{C} = 16\int_{A}^{B}\mathrm{d}z(z^2-A^2)(B^2-z^2)(B^2-A^2)^{-3}f(z)\,z^{-\frac{14}{3}}. \tag{8d}$$

According to (4b) $f(x)$ is a monotonically decreasing function of x when x is not small, e.g. when $x>0.5$, then from (8b, c) we have

$$0 < \tilde{C} < U(A), \tag{9a}$$

$$U(A) = f(A)\,A^{-\frac{17}{3}}\int_{-1}^{1}\mathrm{d}Y(1-Y^2) = \tfrac{4}{3}f(A)\,A^{-\frac{17}{3}}. \tag{9b}$$

Here $U(A)$ is the upper bound of $\tilde{C}$ and approaches zero exponentially for large A; for example, $U(A=5)=2.55\times10^{-23}$ and $U(A=10)=2.61\times10^{-55}$. Hence when A is large, $\tilde{C}$ is nearly zero or is simply equal to zero for a numerical computation on a computer. As will be explained later, this particular character of the integral $C(k,p)$ means that the contribution of the triad interaction (k,p,r) with $A=|k-p|/k_{\mathrm{d}}>\eta$ to the spectrum dynamics in the viscous–convective range can be neglected and η is a numerical constant of order one. Suppose that $\tilde{C}^{(0)}$ is a good approximation of $\tilde{C}$ when $A\leqslant\eta$ and is nearly zero when $A>\eta$, then $\tilde{C}^{(0)}$ will be a good approximation of $\tilde{C}$ for all values of A.

The form of $\tilde{C}^{(0)}$ for the viscous–convective range can be obtained by the asymptotic method. The idealized model of the viscous–convective range corresponds to the limit case $k/k_{\mathrm{d}}\to\infty$ and $k/k_{\mathrm{c}}\to0$, and the ratio of the Corrsin wavenumber k_{c} to the Kolmogorov wavenumber k_{d} approaches infinity. In this idealized case we can make an asymptotic expansion of $\tilde{C}$ for $A\leqslant\eta$ as $\tilde{K}=k/k_{\mathrm{d}}\to\infty$ which implies $\tilde{P}=p/k_{\mathrm{d}}\to\infty$, and from (8$d$) we obtain

$$\left.\begin{aligned} \tilde{C} &= \tilde{C}^{(0)}\{1+O(\tilde{K}^{-2})\} \quad (\tilde{K}\to\infty, A\leqslant\eta), \\ \tilde{C}^{(0)} &= (\tilde{K}\tilde{P})^{-2}G(A), \quad A = |k-p|/k_{\mathrm{d}}, \end{aligned}\right\} \tag{10a}$$

with

$$G(x) = \int_{x}^{\infty}\mathrm{d}z(z^2-x^2)\,f(z)\,z^{-\frac{14}{3}}. \tag{10b}$$

Here x is the independent variable in the definition of a function, later it is also used as a dummy variable in an integral, but its meaning is clear from its context. When higher-order terms $O(\tilde{K}^{-2})$ in the above asymptotic expansion are neglected, we have

$$\tilde{C} \approx (\tilde{K}\tilde{P})^{-2}G(A). \tag{10c}$$

The table of $G(x)$ value is given in the Appendix; $G(x)$ approaches zero exponentially when x is large, so $\tilde{C}^{(0)}$ is nearly zero for large A. Hence (10c) represents a good approximation of $\tilde{C}$ in the sense explained above. It is easy to prove that $G(x) \approx (18/55)\, x^{-\frac{5}{3}}$ as $x \to 0$, so $G(x) \to \infty$ as $x \to 0$. From (8a) and (10c), finally we have

$$C(k,p) = (4\pi)^{-1}\epsilon^{\frac{2}{3}} k_{\mathrm{d}}^{-\frac{5}{3}} k^{-2} p^{-2} G(A), \quad A = |k-p|/k_{\mathrm{d}}. \tag{10d}$$

4. Variance equation

Substituting (10d) into (5b), we obtain

$$S(k) = 4\pi\epsilon^{\frac{2}{3}} k_{\mathrm{d}}^{-\frac{5}{3}} \int_0^\infty \mathrm{d}p\, G(A)\, k^2 p^2 \frac{g(p)-g(k)}{\Omega(k)+\Omega(p)}. \tag{11}$$

When k is in the convective range of stationary turbulence, $d_t F(k) = 0$, the variance equation (5a) becomes

$$\int_k^\infty S(r)\,\mathrm{d}r = 2\mu \int_k^\infty r^2 F(r)\,\mathrm{d}r = 2\mu \int_0^\infty r^2 F(r)\,\mathrm{d}r = \chi, \tag{12}$$

which means that in the convective range the scalar-variance transfer function $\int_k^\infty S(r)\,\mathrm{d}r$ is independent of k and is equal to the dissipation rate of scalar variance. Since the integrand in (11) changes sign when k and p are interchanged, from (11) and (12) we obtain

$$\chi = 4\pi\epsilon^{\frac{2}{3}} k_{\mathrm{d}}^{-\frac{5}{3}} \int_k^\infty \mathrm{d}r \int_0^k \mathrm{d}p\, G(|r-p|/k_{\mathrm{d}})\, r^2 p^2 \frac{g(p)-g(r)}{\Omega(r)+\Omega(p)}. \tag{13}$$

Suppose that $g(k)$ and $\Omega(k)$ have the form of a power function,

$$g(k) = Mk^m, \quad \Omega(k) = Nk^n. \tag{14}$$

Letting $r = ku$ and $p = kv$, and substituting (14) into (13), we have

$$\chi = 4\pi\epsilon^{\frac{2}{3}} k_{\mathrm{d}}^{-\frac{5}{3}} k^{6+m-n} (M/N)\, W, \tag{15}$$

$$W = \int_1^\infty \mathrm{d}u \int_0^1 \mathrm{d}v\, G(\tilde{K}|u-v|)\, u^2 v^2 \frac{v^m - u^m}{u^n + v^n}. \tag{16a}$$

By making change of dummy variables $u = 1+a$ and $v = 1-b$, (16a) becomes

$$W = \int_0^\infty \mathrm{d}a \int_0^1 \mathrm{d}b\, G(\tilde{K}(a+b))\, u^2 v^2 \frac{v^m - u^m}{u^n + v^n}. \tag{16b}$$

For the idealized model of the viscous–convective range corresponding to $\tilde{K} = k/k_{\mathrm{d}} \to \infty$ and $k/k_{\mathrm{c}} \to 0$, we can make an asymptotic expansion of W as $\tilde{K} \to \infty$. Since $G(x)$ decreases to zero exponentially when x is large, as shown in the Appendix, by neglecting higher-order terms in the asymptotic expansion and using the relationship

$$\int_0^\infty \mathrm{d}a \int_0^1 \mathrm{d}b\, G(\tilde{K}(a+b))\,(\ldots) \approx \int_0^\infty \mathrm{d}a \int_0^\infty \mathrm{d}b\, G(\tilde{K}(a+b))\,(\ldots) \quad (\tilde{K} \to \infty),$$

from (16b) we obtain

$$W \approx \int_0^\infty \mathrm{d}a \int_0^\infty \mathrm{d}b\, G(\tilde{K}(a+b))\,(-\tfrac{1}{2}m)\,(a+b) = (-\tfrac{1}{2}m)\,\beta\tilde{K}^{-3} \quad (\tilde{K} \to \infty), \tag{16c}$$

where β is a numerical constant, and by changing dummy variables

$$\beta = \int_0^\infty \mathrm{d}\tilde{A} \int_0^\infty \mathrm{d}\tilde{B}(\tilde{A}+\tilde{B})\, G(\tilde{A}+\tilde{B}), \quad \tilde{A} = \tilde{K}a, \quad \tilde{B} = \tilde{K}b,$$

which can be simplified and becomes

$$\beta = \int_0^\infty \mathrm{d}x\, x^2 G(x). \tag{17}$$

From (15) and 16c) we obtain

$$\chi = 2\pi(-m)\,\beta\epsilon^{\frac{2}{3}} k_\mathrm{d}^{\frac{4}{3}} (M/N)\, k^{3+m-n},$$

where k is a variable but all other quantities are constants; hence

$$3+m-n = 0 \tag{18}$$

and

$$M = -\frac{N\chi\nu}{2\pi m\beta\epsilon}. \tag{19}$$

Here the relationship $\epsilon^{\frac{2}{3}} k_\mathrm{d}^{\frac{4}{3}} = \epsilon/\nu$ has been used.

5. Ω-Equation

Substituting (10d) into the Ω-equation (7), we obtain

$$g(k)\,\Omega(k) = \epsilon^{\frac{2}{3}} k_\mathrm{d}^{-\frac{5}{3}} \int_0^\infty \mathrm{d}p\, G(A)\, p^2 \Omega(p) \frac{g(k)-g(p)}{[\Omega(k)+\Omega(p)]^2}. \tag{20}$$

Letting $p = k(1+y)$, we have $A = |k-p|/k_\mathrm{d} = \tilde{K}|y|$, and it can be shown that the asymptotic approximation (10c) is obtained by neglecting higher-order terms $O(y^2)$ as $\tilde{K} = k/k_\mathrm{d} \to \infty$, while $A = O(1)$, corresponding to the idealized model of viscous–convective range. The omission of higher-order terms $O(y^2)$ implies that

$$g(p) = g(k)\{1+[kg'(k)/g(k)]\,y\}, \quad \Omega(p) = \Omega(k)\left\{1+\left[\frac{k\Omega'(k)}{\Omega(k)}\right]y\right\}. \tag{21}$$

From (14), (20) and (21) we have

$$N^2 k^{2n-3} = \epsilon^{\frac{2}{3}} k_\mathrm{d}^{-\frac{5}{3}} \int_{-1}^\infty \mathrm{d}y\, G(\tilde{K}|y|)\, W(y), \tag{22a}$$

$$W(y) = -\tfrac{1}{4} m y[1+2y+O(y^2)]. \tag{22b}$$

Since $$\int_{-1}^\infty \mathrm{d}y\, G(\tilde{K}|y|)\, W(y) \approx \int_0^\infty \mathrm{d}y\, G(\tilde{K}y)\,[W(y)+W(-y)] \quad (\tilde{K}\to\infty),$$

by using (17), equation (22) becomes

$$N^2 k^{2n-3} = -m\epsilon^{\frac{2}{3}} k_\mathrm{d}^{-\frac{5}{3}} \int_0^\infty \mathrm{d}y\, y^2 G(\tilde{K}y) = -m\epsilon^{\frac{2}{3}} k_\mathrm{d}^{-\frac{5}{3}} \beta \tilde{K}^{-3},$$

or

$$N^2 k^{2n} = -m\beta\epsilon^{\frac{2}{3}} k_\mathrm{d}^{-\frac{4}{3}} = -m\beta\epsilon/\nu. \tag{22c}$$

In (22c) k is a variable but all other quantities are constants, hence

$$n = 0 \tag{23}$$

and

$$N^2 = -m\beta\epsilon/\nu. \tag{24}$$

6. Viscous–convective-range spectrum

From (18) and (23) we obtain
$$m = -3. \tag{25}$$
From (19), (24), and (25), we have
$$M = (2\pi)^{-1}\chi(\nu/\epsilon)^{\frac{1}{2}}(3\beta)^{-\frac{1}{2}}, \tag{26}$$
$$N = (3\beta\epsilon/\nu)^{\frac{1}{2}}. \tag{27}$$
By using (14), (25), and (26), the scalar-variance spectrum is
$$F(k) = 4\pi k^2 g(k) = \chi C(\nu/\epsilon)^{\frac{1}{2}}k^{-1}, \tag{28}$$
$$C = 2(3\beta)^{-\frac{1}{2}}. \tag{29}$$
Equations (28) and (29) correspond to (1) and (3).

Using (4*b*), (10*b*), and (17), a numerical computation gives $\beta = 0.0666$; then from (29) we have $C = 4.47$. There arises the question of whether the values of β and C depend upon the form of the function $f(x)$.

Actually it can be shown that for this model the values of β and C are independent of the concrete forms of the approximate formula for $f(x)$. From (10*b*) and (17) we have
$$\beta = \int_0^\infty \mathrm{d}x\, x^2 \int_x^\infty \mathrm{d}z\,(z^2-x^2)f(z)z^{-\frac{14}{3}} = \tfrac{2}{15}\int_0^\infty \mathrm{d}z\, z^{\frac{1}{3}}f(z). \tag{30}$$
From the definition of the energy dissipation rate, using (4*a*), we have
$$\epsilon = 2\nu \int_0^\infty \mathrm{d}k\, k^2 E(k) = 2\nu\epsilon^{\frac{2}{3}}\int_0^\infty \mathrm{d}k\, k^{\frac{1}{3}}f(k/k_{\mathrm{d}}),$$
or
$$2\int_0^\infty \mathrm{d}x\, x^{\frac{1}{3}}f(x) = 1. \tag{31}$$
From (30) and (31) we have
$$\beta = \tfrac{1}{15}. \tag{32}$$
Substituting (32) into (29), we obtain
$$C = 2\sqrt{5} \approx 4.47. \tag{33}$$
From (14), (23), (27), and (32), we have
$$\Omega(k) = N = (1/\sqrt{5})\,(\epsilon/\nu)^{\frac{1}{2}}. \tag{34}$$

7. Discussion

In previous work the method of non-equilibrium statistical mechanics combined with a perturbation-variation approach has been applied to derive the velocity spectrum in the universal range and the scalar-variance spectrum in the inertial–convective range, and the universal constants predicted from these spectra functions agree with experimental data (Qian 1983, 1984, 1985). In this paper the same method is applied to the study of the spectrum dynamics of a turbulent passive scalar field in the viscous–convective range. Batchelor's k^{-1} spectrum is derived, and the predicted universal constant $C = 2\sqrt{5}$ is in agreement with experimental data $C = 3.9 \pm 1.5$ (*Grant et al.* 1968).

The results of this paper further confirm the conclusion of Kraichnan's analysis that the actual value of the universal constant C would be greater than the values suggested by Batchelor (1959) and Gibson (1968) who have neglected the statistical character of the least principal rate of strain. The prediction $C = 2\sqrt{5}$ obtained in this paper is also much better than the prediction $C < 0.9$ of Kraichnan's ALHDI approximation (Kraichnan 1968; Leslie 1973). By proper choice of the empirical constants, the TFM (Newman & Herring 1979) or the EDQNM approximation (Herring *et al.* 1982) might produce a result which is better than Kraichnan's ALHDI estimate but still less than the experimental data of Grant *et al.* (1968). The prediction made in this paper, like the prediction by Kraichnan's DIA (or LHDIA) theory, is a theoretical prediction from a self-consistent analytical method free of adjustable empirical constants, and is in agreement with experimental data.

According to the spectrum-dynamical equations (5)–(7), the elementary process of the spectrum dynamics of a passive scalar field convected by turbulence is the triad interaction (k, p, r). Here wavenumbers k and p are connected with the scalar spectrum while the wavenumber r is connected with the velocity spectrum, and $A \leqslant r/k_d \leqslant B$. Since $f(x)$ approaches zero exponentially, the major contribution to the spectrum dynamics is made by the wavenumber $r \approx Ak_d$, the contribution of the wavenumber $r \gg Ak_d$ is minor. Let

$$\xi(x) = \beta^{-1} \int_0^x \mathrm{d}z\, z^2 G(z), \tag{35}$$

then $\xi(\infty) = 1$ by (17). From the analysis developed in §§3–6, $\xi(x)$ represents the percentage of the total contribution made by the triad interactions $(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{r})$ with $A = |k-p|/k_d \leqslant x$. The table of $\xi(x)$ values given in the Appendix shows that 99.9% of the total contribution is made by the triad interaction (k, p, r) with $A \leqslant 1$. Hence the following statement, made in §3, is further justified: the contribution of the triad interaction (k, p, r) with $A = |k-p|/k_d > \eta$ can be neglected and η is a numerical constant of order one.

The values of $G(x)$ and $\xi(x)$ in the Appendix are obtained by using (4*b*). A well-known semi-empirical formula for $f(x)$ is (Pao 1965)

$$f(x) = Ko \exp\left(-1.5 Ko\, x^{\frac{4}{3}}\right), \quad Ko = 1.7. \tag{36}$$

A calculation of $G(x)$ and $\xi(x)$ was also made by using Pao's formula (36), and the resulting $G(x)$ and $\xi(x)$ have the same behaviour as in the Appendix, although the numerical values are not identical. This behaviour enables us to simplify the integral $C(k, p)$, the variance equation, and the Ω-equation by an asymptotic method.

An interesting corollary of the behaviour of $\xi(x)$ given in the Appendix is that the most important wavenumber interval of the velocity spectrum is the interval $[0.1k_d, k_d]$ instead of the one which is far away from k_d. Eddies with most of their energy in the interval $[0.1k_d, k_d]$ are the eddies which make the major contribution to the spectrum dynamics in the viscous–convective range, and will be called 'major eddies'. These major eddies have a characteristic length $l \approx \eta/k_d$ and a characteristic time $\tau \approx \eta(\nu/\epsilon)^{\frac{1}{2}}$ (η is a numerical constant of order one). Hence these major eddies give rise to an effective damping coefficient $\Omega(k) \approx \tau^{-1} \approx (\epsilon/\nu)^{\frac{1}{2}}/\eta$, which is in agreement with (34). In the viscous–convective range, owing to $f(x)$ decreasing exponentially, these major eddies are related to the wavenumbers of the velocity spectrum, which are independent of the wavenumber k of the scalar spectrum, so $\Omega(k) \approx \tau^{-1}$ is a constant independent of k, as shown by (34). However, in the inertial–convective range the effective damping coefficient $\Omega(k) = 0.5\epsilon^{\frac{1}{3}} k^{\frac{2}{3}}$ increases with k (Qian 1985), because in this case the eddies which make the major contribution are related to the wavenumbers of the velocity spectrum which increases with the wavenumber k of the scalar spectrum, so their characteristic time decreases with k, and $\Omega(k) \approx \tau^{-1}$ increases with k.

Strictly speaking, the derivation and results of this paper are valid only for the idealized model of the viscous–convective range corresponding to the limit case $k/k_d \to \infty$ and $k/k_c \to 0$. In this idealized limit case the integral $C(k,p)$, the variance equation, and the Ω-equation can be simplified by an asymptotic expansion so that an analytical solution of the complicated integral equations (5) and (7) is possible. The results of this paper are approximately valid for a real turbulent flow with large k_c/k_d.

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Appendix. Table of $G(x)$ and $\xi(x)$ values

x	$G(x)$	$\xi(x)$
0.0001	0.1849×10^{7}	0.206×10^{-4}
0.001	0.4295×10^{5}	0.468×10^{-3}
0.01	0.1157×10^{4}	0.0121
0.02	393.7	0.0328
0.04	127.5	0.0880
0.06	62.42	0.153
0.08	36.04	0.220
0.1	22.75	0.291
0.2	4.063	0.589
0.3	1.065	0.780
0.4	0.3245	0.888
0.5	0.1066	0.945
0.6	0.03653	0.974
0.7	0.01284	0.988
0.8	0.004579	0.995
0.9	0.001646	0.998
1	0.5942×10^{-3}	0.999
2	0.1816×10^{-7}	1.000
3	0.2670×10^{-12}	1.000
4	0.1770×10^{-17}	1.000
5	0.5569×10^{-23}	1.000
6	0.8823×10^{-29}	1.000
7	0.7432×10^{-35}	1.000
8	0.3487×10^{-41}	1.000
9	0.9482×10^{-48}	1.000
10	0.1546×10^{-54}	1.000

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